JOURNAL OF APPROXIMATION THEORY 38, 334–337 (1983)

Existence of Transformed Rational Complex Chebyshev Approximations, II*

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Communicated by Oved Shisha

Received January 9, 1978, revised June 16, 1980

This paper is a continuation of the author's existence theory [4]. The problem is the same except that a different convention for defining approximations is also considered and it is assumed that $|\sigma(t)| \to \infty$ as $|t| \to \infty$. Readers should consult [4] for the problem statement, notation and preliminaries.

We need a convention for defining approximations $F(A, \cdot)$ where the denominator $Q(A, \cdot)$ vanishes. We will adapt one due to Goldstein [2; 3, 85ff].

Convention. Let $Q(A, x) = 0$. If $P(A, x) \neq 0$, $F(A, x) = \sigma(\infty)$. If $P(A, x) = 0$, $F(A, x) = f(x)$.

While this convention can be applied to all X, it is most appropriate for finite X, as no other convention seems practical. In particular, even if we could appply Boehm's convention [4], we would not necessarily get existence—in Example 1, below, an approximation zero on $\{\frac{1}{2}, 1\}$ is identically zero with Boehm's convention. If X is infinite, Goldstein's may not be the most satisfactory convention, as the convention may give $F(A, \cdot)$ discontinuities which need not exist with other conventions.

THEOREM. *Let P be a non-empty closed subset of $\hat{P}$. With Goldstein's convention, there exists a best parameter from P to all $f \in C(X)$.*

Proof. The proof is exactly the same as the proof for the corresponding result in [4] up to the case $Q(A, x) \neq 0$.

If $Q(A, x) = 0$ we have two possibilities. First, we might have $P(A, x) \neq 0$ in which case $P(A^k, x)/Q(A^k, x) \to \infty$, $F(A^k, x) \to \infty$, and

* Written while the author was on sabbatical at the University of British Columbia.

0021-9045/83 $3.00

$|f(x) - F(A^k, x)| \to \infty$, which contradicts the choice of A^k. There only remains the possibility that $P(A, x) = 0$, in which case

$$|f(x) - F(A, x)| = 0 \leqslant \rho(f).$$

Three classes of closed sets of coefficients are given in [4]. A fourth follows:

Let $Y = \{y_1, ..., y_s\}$ be a finite subset of X. Let

$$P_t = \{A : A \in \hat{P}, F(A, y_i) = f(y_i), i = 1, ..., s\}.$$

P_t is closed. Assume $\{A^k\} \in P_t$, $\{A^k\} \to A$. Take $y \in Y$. If $Q(A, y) \neq 0$, $F(A^k, y) = f(y) \to F(A, y)$. If $Q(A, y) = 0$ and $P(A, y) \neq 0$, $|F(A^k, y)| \to \infty$, which is a contradiction. If $Q(A, y) = P(A, y) = 0$, $F(A, y) = f(y)$ by convention. It should be noted that P_t may not be closed with Boehm's convention [4].

Admissible Approximation

Dolganov [1] defines a rational function $R(A, \cdot)$ to be *admissible* if $\text{Re}(Q(A, \cdot)) > 0$. Even in approximation by ratios of power polynomials (ordinary rational functions), a best approximation need not exist if we do not permit denominators to have zeros.

Example 1. Let $X = \{0, \frac{1}{2}, 1\}$ and $F(A, x) = a_1/(a_2 + a_3 x)$. Let $f(0) = 1$ and $f(\frac{1}{2}) = f(1) = 0$. As $1/(1 + kx) \to f$ uniformly on X, $\rho(f) = 0$. But if $Q(A, \cdot)$ cannot have zeros, the only approximant vanishing on $\{\frac{1}{2}, 1\}$ is identically zero.

Examples of non-existence on sets with no isolated points are given in [4].

Real Approximation

Consider the case in which all basis functions are real, all coefficients are real, σ is a continuous mapping of the real line into the extended real line, and f is real. This is the case of real Chebyshev approximation by transformed rational functions. A special case is where $\sigma(x) = x$, which has already been studied by Goldstein. The existence theorem obtained earlier in this paper applies. Of particular interest is the case where P is the subset of $\hat{P}$ with $Q(A, \cdot) \geqslant 0$. This is P_r with K being the non-negative real line.

Approximation with Unbounded Bases

In this and following sections we drop the requirement that X be compact but still require that basis functions be continuous on X. A problem of this type with all basis functions *bounded* has already been considered by Boehm [5]. We consider the case in which basis functions may be unbounded on X. A classical case where this happens is when X is an unbounded subset of the real line or complex plane, for example, $[0, \infty)$ or $(-\infty, \infty)$, and we approximate by ordinary rational functions

$$R_m^n = \{p/q : q \in H_n, q \in H_m, q \not\equiv 0\}, \tag{1}$$

where H_l is the set of power polynomials of degree l. We claim that the existence theorem of the previous paper [4] and of this paper holds for the case of unbounded basis functions providing f is bounded on X (in addition to being continuous).

If all approximants are unbounded, $\rho(f) = \infty$ and we have existence trivially. If at least one approximant is bounded, $\rho(f) < \infty$ and we follow the existence proof of [4] down to an inequality involving $|R(A, x)|$ on the left-hand side. We replace that inequality by the following discussion. By the normalization $\sum_{k=1}^{m} |a_{n+k}| = 1$,

$$|Q(A, x)| \leqslant \sum_{k=1}^{m} |\psi_k(x)|. \tag{2}$$

If the right-hand side of (2) is >0, we can write the inequality

$$|R(A, x)| = |P(A, x)|/|Q(A, x)| \geqslant |P(A, x)| \Big/ \sum_{k=1}^{m} |\psi_k(x)|. \tag{3}$$

If the right-hand side of (2) is zero, we have

$$|R(A, x)| < \infty \rightarrow P(A, x) = 0. \tag{3'}$$

Let Y be any n-point subset on which $\{\phi_1, ..., \phi_n\}$ is linearly independent. Let there exist M such that

$$\max \{|R(A^k, x)| : x \in Y\} < M; \tag{4}$$

then (4) implies by (3), (3′) that

$$\max \{|P(A^k, x)| : x \in Y\}$$

is bounded. This implies that the numerator coefficients of the sequence $\{A^k\}$ are bounded and we use the rest of the proof of [4] and its modification for Goldstein's convention earlier in this paper.

Ordinary Rational Approximations

Consider approximation by R_m^n (defined by (1)) on X, a subset of the complex plane or real line. First, let us assume X has no isolated points and use Boehm's convention. The existence theorem of [4] applies and we have existence of a best approximation p/q. By considering multiplicities of zeros of polynomials, it can be seen by standard arguments that p/q can be replaced by p_0/q_0, p_0 and q_0 relatively prime and q_0 having no zeros on $\bar{X}$, the closure of X.

Remark. If we approximate with constraints, the above may not be true. For example, we might want denominators $\geqslant 0$. In the case $X = \{x : |x| \geqslant 1\}$, x/x^2 has a positive denominator on X but removing common factors gives $1/x$.

In the case of real approximation and X an interval, q_0 is of one sign on $\bar{X}$; hence (by changing the sign of p_0 and q_0 if necessary) we can assume $q_0 > 0$ and there is a best approximation which is admissible (that is, its denominator is >0). The same results hold if we transform R_m^n by σ. Second, let us apply Goldstein's convention. The existence theorem of this paper applies and we get existence of a best approximation p/q. However, we cannot always cancel out common factors (see Example 1). Existence remains if we transform R_m^n by σ.

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